THE THERMOCAPILLARY EFFECT AND SMALL OSCILLATIONS OF A LIQUID AT LARGE MARANGONI NUMBERS

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1. We consider the nonlinear problem of small oscillations of a viscous, heat-conducting liquid in an unbounded region D under thermocapillary forces generated by nonequilibrium heating of the free boundary Γ. We consider the limit of small viscosity ($\nu \to 0$) and small thermal diffusivity ($\chi \to 0$). The equations of the problem are

$$\partial \mathbf{v}/\partial t + (\mathbf{v}, \nabla)\mathbf{v} = -\rho^{-1}\nabla p + \nu\Delta\mathbf{v} + \mathbf{g},$$
$$\partial T/\partial t + \mathbf{v}\nabla T = \chi\Delta T, \ \operatorname{div} \mathbf{v} = 0; \tag{1.1}$$

$$2\nu\rho\Pi\mathbf{n} - 2\nu\rho(\mathbf{n}\Pi\mathbf{n})\mathbf{n} = \nabla_\Gamma\sigma, \ T = T_\Gamma, \ (x, y, z) \in \Gamma,$$
$$p = 2\nu\rho\mathbf{n}\Pi\mathbf{n} + \sigma(k_1 + k_2) + p_*, \quad \partial F/\partial t + \mathbf{v}\nabla F = 0, \ (x, y, z) \in \Gamma. \tag{1.2}$$

Here $\mathbf{v} = (v_x, v_y, v_z)$; T is the temperature $\mathbf{g} = -g\mathbf{e}_z$; $\mathbf{e}_z = (0, 0, 1)$ is a unit vector along the z axis; g is the acceleration of gravity; $\mathbf{n}$ is the outward unit normal to the free surface Γ; Π is the deformation rate tensor; k_1, k_2 are the principal curvatures of the surface Γ; $p_* =$ const is the pressure on Γ; $\nabla_\Gamma = \nabla - (\mathbf{n}\nabla)\mathbf{n}$ is the gradient along Γ; $F(t, x, y, z) = 0$ is the equation of the free surface in implict form; the surface tension σ is assumed to be a linear function of temperature $\sigma = \sigma_0 + \sigma_T (T - T_*)$ (σ_0, σ_T, T_* are known constants and $\sigma_T < 0$). The velocity field and temperature gradient vanish at infinity. Initial conditions are not specified, since the solutions will be constructed in the form of free oscillations.

A nonlinear boundary layer is formed near the free surface in the limit of zero viscosity and zero thermal conductivity. In the unbounded region outside the boundary layer the flow of the liquid is described approximately by Euler's equations. Nonlinear Marangoni boundary layers near a free surface formed as a result of the thermocapillary effect were studied in [1-4]. An asymptotic expansion in the limit $\nu \to 0$ was obtained in [5] for the steady flow of an incompressible fluid subject to nonequilibrium heating of the free surface.

Formal asymptotic expansions of the solution of the problem (1.1), (1.2) in the limit $\nu, \chi \to 0$ will be constructed below. The problem is converted to dimensionless form with the small parameter $\varepsilon = M^{1/3}$ ($M = |\sigma_T|L^2A\rho^{-1}\nu^{-2}$ is the Marangoni number and L and A are the characteristic scales of length and temperature gradient). We note that small ε corresponds to small ν or large temperature gradient. The dimensionless pressure p' is defined by the relation $p = Pp' - \rho gz$ ($P = A|\sigma_T|$ is the scale of pressure). The typical velocity $U = (\sigma_T{}^2A^2L\nu^{-1}\cdot\rho^{-2})^{1/3}$ in the boundary layer near the free surface is used as the scale of velocity. The quantity $(L/g)^{1/2}$ is used as the scale of time. The asymptotic expansion of the solution of (1.1), (1.2) is constructed in the form

$$\mathbf{v} \sim \mathbf{h}_0 + \varepsilon^{1/2}(\mathbf{v}_1 + \mathbf{h}_1) + \ldots, \ p' \sim q_0 + p_0 + \varepsilon(p_1 + q_1) + \ldots,$$
$$T \sim \theta_0 + T_0 + O(\varepsilon^{1/2}), \ \zeta \sim \zeta_0 + \varepsilon^{1/2}\zeta_1 + \ldots, \tag{1.3}$$

where $z = \zeta(x, y, t)$ is the equation of the free surface. Let D_Γ denote the boundary layer. Then $\mathbf{h}_k$, q_k, θ_0 are solutions of the boundary-layer problem in D_Γ and $\mathbf{v}_1$, p_0, p_1, T_0 define the solution outside D_Γ. The orders of magnitude of the leading terms in the expansions (1.3) are found by assuming that the viscous and inertial terms in the Navier-Stokes equations and in the boundary conditions for the tangential stresses are of the same order of magnitude. Then the thickness of the boundary layer is of order ε.

Note. The asymptotic solution of (1.1), (1.2) was constructed in [6] in the linear case where the temperature dependence is not taken into account and the tangential stress on Γ is given. In both the linear and nonlinear problems, when the surface tangential stress

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is finite ($\nabla\Gamma\sigma = O(1)$ in the limit $\nu \to 0$) the velocity of the liquid in the boundary layer near the free surface is an order of magnitude larger than in the outer flow [see Eq. (1.3)]. Inclusion of the nonlinearity changes the thickness of the boundary layer δ_p: in the linear case $\delta_p \sim \nu^{1/2}$, while in the nonlinear case $\delta_p \sim \nu^{2/3}$.

2. A boundary-value problem for the leading terms of the asymptotic expansion (1.3) determining the flow in the boundary layer D_Γ is obtained by a second to (1.1), (1.2) using the Vishik–Lyusternik method [7]. Near the free surface we introduce the moving local orthogonal coordinates ξ, φ, θ (ξ is the distance between the point M and the surface Γ and φ and θ are the curvilinear coordinates of the projection of M onto Γ). The surfaces $\varphi = c_1(t)$, $\theta = c_2(t)$ form two families of orthogonal surfaces chosen so that their lines of intersection with Γ are lines of principal curvature. It is assumed that for sufficiently small ξ the segments normal to Γ do not intersect one another. The transformation from Cartesian to moving local coordinates is given by

$$\mathbf{r} = \mathbf{R}(t, \varphi, \theta) + \xi\mathbf{n}(t, \varphi, \theta),$$

where $\mathbf{r} = (x, y, z)$; $\mathbf{R} = (X, Y, Z)$. We note that $\mathbf{r} = \mathbf{R}(t, \varphi, \theta)$ is the parametric equation of the surface Γ, which deforms in time. The local coordinate system moves with the surface Γ.

The equations (1.1) and the boundary conditions (1.2) are transformed to the moving coordinates. We introduce the "fast" time t_1 associated with the small viscosity $t_1 = t/\sqrt{\varepsilon}$ and use the fact that the functions $\mathbf{h}_k$, q_k depend on two different time scales: t and t_1. An expression for the pressure inside the boundary layer is obtained by applying the Vishik–Lyusternik method to (1.1):

$$q_0 = -k_1\int_s^\infty h_{\varphi 0}^2 ds - k_2\int_s^\infty h_{\theta 0}^2 ds + \int_s^\infty\left[\left(H_\varphi^{-1}\frac{\partial^2\xi}{\partial t\partial\varphi} - \omega_1\right)h_{\varphi 0} + \left(H_\theta^{-1}\frac{\partial^2\xi}{\partial t\partial\theta} - \omega_2\right)h_{\theta 0}\right]ds. \quad (2.1)$$

Here H_φ and H_θ are the Lamé coefficients of the surface Γ; ω_1 and ω_2 are given by

$$\omega_1 = k_1 H_\varphi\left(\frac{\partial\varphi}{\partial t} - \frac{\partial\varphi}{\partial t}\Big|_{\xi=0}\right), \quad \omega_2 = k_2 H_\theta\left(\frac{\partial\theta}{\partial t} - \frac{\partial\theta}{\partial t}\Big|_{\xi=0}\right).$$

We assume that Γ oscillates about the stationary surface Γ_c with an amplitude of order ε and a velocity $O(\varepsilon)$. Then $\partial\varphi/\partial t$, $\partial\theta/\partial t$, $\partial\xi/\partial t$ are of order ε and to $O(\varepsilon)$ (2.1) becomes

$$q_0 = -k_1\int_s^\infty h_{\varphi 0}^2 ds - k_2\int_s^\infty h_{\theta 0}^2 ds. \quad (2.2)$$

Taking the coordinate φ to be the arc length along the free surface, the boundary-layer equations for the plane problem are

$$\begin{gathered}\frac{\partial h_{\varphi 0}}{\partial t_1} + H_{\xi 2}\frac{\partial h_{\varphi 0}}{\partial s} + h_{\varphi 0}\frac{\partial h_{\varphi 0}}{\partial\varphi} = \frac{\partial^2 h_{\varphi 0}}{\partial s^2}, \quad \frac{\partial h_{\varphi 0}}{\partial\varphi} + \frac{\partial H_{\xi 2}}{\partial s} = 0,\\ \frac{\partial h_{\varphi 0}}{\partial s} = -\frac{\partial\sigma(t, \varphi)}{\partial\varphi}, \quad H_{\xi 2} = 0 \ (s = 0),\\ h_{\varphi 0} = h_{\xi 2} = 0 \ (s = \infty), \ H_{\xi 2} = h_{\xi 2} + \mathbf{v}_2\mathbf{n}|_\Gamma.\end{gathered} \quad (2.3)$$

Here $s = \xi/\varepsilon$ is the dilated variable, $h_{\varphi 0}$ and $h_{\xi 2}$ are the longitudinal and transverse components of the velocity vector inside the boundary layer, and $h_{\xi 0} = 0$.

Since the surface tension σ depends only on φ and the slow time t, which is independent of the viscosity, we consider solutions of (2.3) which are independent of the fast time t_1, i.e., we assume $h_{\varphi 0} = h_{\varphi 0}(s, \varphi, t)$. It was shown in [4] that the boundary-value problem obtained from (2.3) with $\partial h_{\varphi 0}/\partial t_1 = 0$ and with t as a parameter has a unique solution. Hence the vector function $\mathbf{h}_0(s, \varphi, t)$ determines the velocity field in the quasisteady boundary layer.

3. The functions $\mathbf{v}_1$, p_0, ζ_0 determining the inviscid flow outside the boundary layer and the asymptotic form of the free surface are obtained by a first iteration [7] to (1.1), (1.2). Let Γ_0 be the free surface of the inviscid flow. Near Γ_0 we introduce the moving

local orthogonal coordinates ξ_1, φ_1, θ_1 (ξ_1 is the distance to Γ_0). The principal curvatures of the surface Γ are written in series form $k_i = k_{i0} + \varepsilon^{1/2}k_{i1} + \ldots$ $(i = 1;\ 2)$ (k_{i0} are the principal curvatures of the surface Γ_0). Substituting the expansion (1.3) into (1.1), (1.2), using the fact that $\mathbf{h}_0 = \mathbf{h}_1 = q_0 = q_1 - \theta_0 = 0$ outside the boundary layer and equating the coefficients of ε^0, and ε to zero, we obtain the following boundary-value problem for $\mathbf{v}_1$, p_0, ζ_0 in dimensional form

$$\begin{gathered}\frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_1, \nabla)\mathbf{v}_1 = -\rho^{-1}\nabla p_1, \quad \operatorname{div}\mathbf{v}_1 = 0, \\ p_0 = \rho g\zeta_0 + \sigma(k_{10} + k_{20}) + q_0|_{s=0} + p_*, \quad (x, y, z) \in \Gamma_0, \\ \frac{\partial \zeta_0}{\partial t} + v_{x0}\frac{\partial \zeta_0}{\partial x} + v_{y0}\frac{\partial \zeta_0}{\partial y} = v_{z0} \qquad (z = \zeta_0).\end{gathered} \tag{3.1}$$

As before, initial conditions are not specified for (3.1), since we will be interested only in the small oscillations defined by (3.1).

Tangential stress on the free surface of a low-viscosity liquid therefore leads to the additional term q_0 [see Eq. (2.2)] in the dynamical boundary condition of (3.1) for limiting inviscid flow. This term depends on the velocity field in the boundary layer, the principal curvatures, and the tangential loads.

4. We consider small oscillations about a steady solution. The steady motion of a liquid under nonequilibrium heating of its free surface at large Marangoni numbers has been considered in [5]. The equation for the free surface is

$$(k_1 + k_2)\sigma + k_1\int_0^\infty h_{\varphi 0}^2 ds + k_2\int_0^\infty h_{\theta 0}^2 ds = \rho g z + c. \tag{4.1}$$

The equations of motion for small oscillations in the case of unsteady heating are obtained by linearizing the boundary-value problem (3.1). The equation for the free surface is obtained by linearizing about the surface (4.1). Examples were given in [5] of the calculation of the free surface Γ_c using (4.1). Let Γ_t be the time-dependent surface close to Γ_c and let N be the distance along the normal to Γ_c. Then in local coordinates $N = N(\varphi, \theta, t)$ is the equation of Γ_t (φ, θ parametrizes the surface Γ_c at time t). We linearize the dynamical boundary condition in (3.1) using the relations between the curvatures and arc lengths along the principal directions for the close surface [8]. Because of its complexity, the problem of small oscillations near a three-dimensional curved surface (4.1) is not considered here. We assume that the external inviscid flow is irrotational. Then defining the velocity potential Φ by the equation $\mathbf{v}_1 = \nabla\Phi$ we find from (3.1) that Φ satisfies Laplace's equation and the pressure p_1 is easily eliminated from (3.1). The boundary-value problem describing the small oscillations of the liquid near a plane surface Γ_c can be written in the dimensional form

$$\begin{gathered}\Delta\Phi = 0, \\ -\rho\frac{\partial \Phi}{\partial t} - \rho g N + \sigma\Delta N + \frac{\partial^2 N}{\partial x^2}\int_0^\infty h_{\varphi 0}^2 ds + \frac{\partial^2 N}{\partial y^2}\int_0^\infty h_{\theta 0}^2 ds = 0 \ (z = 0), \\ \frac{\partial \Phi}{\partial n} = \frac{\partial N}{\partial t} \ (z = 0), \ \nabla\Phi = 0 \ (x^2 + y^2 + z^2 = \infty).\end{gathered} \tag{4.2}$$

We consider the plane problem for the oscillations of the liquid near a horizontal surface ($z = 0$) for nonequilibrium heating given by $T = T_* + ALf(x, t)$, where $f(x, t)$ is the dimensionless temperature. The surface tension is then $\sigma = \sigma_0(1 - \lambda f(x, t))$, where $\lambda = |\sigma_T|AL/\sigma_0 \geq 0$. Considering σ to be positive, we assume that for certain t the coefficient σ can vanish at the point of maximum temperature. This leads to the condition $0 \leqslant \lambda \leqslant \left(\max\limits_{x,t} f\right)^{-1}$. The following relation is analogous to a relation given in [5] and is valid in the case considered here:

$$\int_0^\infty h_{\varphi 0}^2 ds = \sigma(x, t) - \sigma(x_0, t) + \int_0^\infty f_0^2 ds$$

($f_0 = h_{\varphi 0}|_{x=x_0}$ is the velocity profile at $x = x_0$). Then the dynamical boundary condition of (4.2) can be written in the plane case as

$$\rho \frac{\partial \Phi}{\partial t} + \rho g \zeta_0 - \sigma_0 (1 + \lambda f(x_0, t) - 2\lambda f(x, t)) \frac{\partial^2 \zeta_0}{\partial x^2} = 0. \tag{4.3}$$

Here we have taken the origin of the coordinate system at the point of maximum temperature where $f_0 = 0$. The latter relation is easily proven, since near the maximum a self-similar solution for the Marangoni boundary layer can be constructed [2].

We further assume that the free surface is heated locally and therefore f can be written as $f = f(x/\delta, t)$($\delta \ll 1$ is a small parameter). Using (4.3), we average (4.2) over the x coordinate [9]. We introduce the slow variable $x_1 = x/\delta$ and expand the functions Φ and ζ_0 in power series in the parameter δ. The leading terms of the asymptotic expansions satisfy (4.2), in which the dynamical boundary condition is replaced by

$$\rho \frac{\partial \Phi}{\partial t} + \rho g \zeta_0 - \sigma_0 (1 + \lambda f_c(t)) \frac{\partial^2 \zeta_0}{\partial x^2} = 0,$$

where $f_c = f(0, t)$ is the temperature at the maximum. Note that this last equation is also obtained when the temperature of the free surface is a delta function. Note also that the boundary-layer equations have the exact solution $h_{x0} = 6k_t^2 x^{-1/3} \mathrm{ch}^{-2}(k_t s x^{-1/3})$ (k_t is a parameter).

Assuming that the liquid occupies the half space $z \leqslant \zeta(x, t)$, $x \in (-\infty, \infty)$, the solution can be found explicitly

$$\Phi = B'(t) \exp(kz) \cos(kx), \quad \zeta_0 = kB(t) \cos(kx).$$

The oscillation amplitude satisfies the equation

$$B'' + [gk + k^3 \sigma_0 \rho^{-1}(1 + \lambda f_c(t))]B = 0 \tag{4.4}$$

(k is the wave number).

Assuming that the heating of the free surface does not depend on time ($f_c \equiv 1$), (4.4) describes harmonic oscillations $B = B_0 \exp(i\omega t)$ with frequency

$$\omega = \sqrt{gk + k^3 \sigma_0 (1 + \lambda) \rho^{-1}}.$$

In this case the thermocapillary effect leads to an increase in the oscillation frequency.

If the temperature of the free surface increases linearly with time $f_c = 1 + bt$, then the solution can be expressed in terms of the Bessel functions $J_{1/3}$, $Y_{1/3}$:

$$B = \sqrt{\tau}\left[c_1 J_{1/3}\left(\frac{2}{3}\tau^{3/2}\right) + c_2 Y_{1/3}\left(\frac{2}{3}\tau^{3/2}\right)\right],$$
$$\tau = [\rho g k + (1 + \lambda)k^3 \sigma_0 + \lambda b \sigma_0 t]/(\lambda \rho^{1/2} k^3 \sigma_0).$$

The thermocapillary effect in this case leads to damping of the small oscillations.

We consider small oscillations for the case of local heating with periodically varying temperature $f_c = a - b\cos(2\Omega t)$, where $|b| < a$. Then (4.4) reduces to the well-known Mathieu equation, for which the stable and unstable regions are well known [10]. In this case the oscillation amplitude can diverge, damp out, or oscillate periodically in time, depending on the values of the parameters. We not that the oscillations always increase for $b \neq 0$ and $\lambda > 0$ satisfying the relation $\rho g k + \lambda a k^3 \sigma_0 = \rho \Omega^2 m^2$ ($m = 1, 2, 3, \ldots$).

5. We consider the small oscillations of a liquid in a cylindrical container of infinite depth subject to harmonic heating of the free surface. The surface tension is given by $\sigma = \sigma_0(1 - \lambda f(x))$, $f(x) = \cos(\pi x/l)$ $(0 \leqslant x \leqslant 2l)$, where $0 \leq \lambda \leq 1$. When $\lambda = 1$ the surface tension σ vanishes at $x = 0$ and $x = 2\pi$. Let 2ℓ and $2L$ be the lengths of the sides of the container along the x and y axes. Then the temperature reaches a maximum along the line of contact of the free surface and solid boundary. The problem of small oscillations reduces to Laplace's equation for the potential Φ with the following boundary conditions on the unperturbed horizontal surface Γ_c:

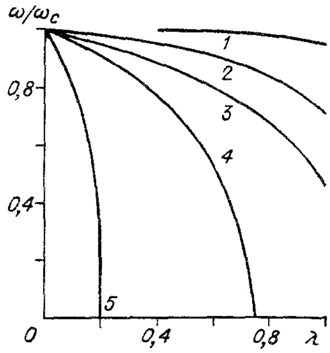

Fig. 1

$$-\rho\frac{\partial\Phi}{\partial t}-\rho g\zeta_0+\sigma_0(1+\lambda F(x))\frac{\partial^2\zeta_0}{\partial x^2}+\sigma_0(1-\lambda f(x))\frac{\partial^2\zeta_0}{\partial x^2}=0,$$

$$\frac{\partial\Phi}{\partial z}=\frac{\partial\zeta_0}{\partial t}\quad(z=0),\ F=1-2f(x).$$

On the solid walls we have the nonpenetration conditions

$$\frac{\partial\Phi}{\partial x}=0\ (x=0,\ x=2l),\ \frac{\partial\Phi}{\partial y}=0\quad(y=0,\ y=2L).$$

The solution is written in Fourier series form

$$\Phi=\cos\frac{\pi ny}{2L}\sum_{m=1}^{\infty}A_m\exp(k_{m,n}z)\cos\frac{\pi mx}{2l}\cos\omega t,$$
$$\zeta_0=\cos\frac{\pi ny}{2L}\sum_{m=1}^{\infty}\frac{k_{m,n}}{\omega}A_m\cos\frac{\pi mx}{2l}\sin\omega t,\ k_{m,n}=\sqrt{\left(\frac{\pi m}{2l}\right)^2+\left(\frac{\pi n}{2L}\right)^2},\ n=0,1,2,\ldots \tag{5.1}$$

The case n = 0 corresponds to oscillations in a channel with walls parallel to the y axis.

In the absence of the temperature gradient ($\lambda = 0$) the natural frequencies can be found explicitly:

$$\omega=\omega_c=\sqrt{gk_{m,n}+\sigma_0k_{m,n}^3/\rho}.$$

In the general case the oscillation frequencies satisfy a transcendental equation and can be computed numerically when $\lambda > 0$. By retaining the first four harmonics in the Fourier series we calculated the frequencies ω to three significant figures. The first few natural frequencies normalized to ω_c are shown in Fig. 1 as functions of the temperature amplitude λ for n = 1 and $\ell = L$. Curves 1-5 correspond to Bond numbers $Bo = \rho g\ell^2/\sigma_0$ equal to 10, 1, 0, −1, −1.8. For fixed Bond number the oscillation frequencies decrease monotonically with increasing temperature. The case Bo = 0 corresponds to zero gravity. As Bo increases the frequencies increase at fixed λ and $\omega \to \omega_c$ when $Bo \to \infty$.

The oscillation frequencies were also calculated in the case $f(x) = \cos(\pi x/\ell)$, $-\ell \le x \le \ell$, i.e., when the temperature reaches a minimum value on the lines of contact $x = \pm\ell$ of the free surface and solid boundary. Now the natural frequencies increase monotonically with increasing temperature amplitude and decrease monotonically with increasing Bo. Heating of the free surface increases the value of ω.

The oscillation frequencies in a rectangular container when the heating of the free surface is a given by a delta function $f(x) = \lambda\delta(x) + \lambda\delta(\ell - x)$ $(0 \le r \le \ell)$ are given by

$$\omega^2=gk_{m,n}\left[1+\left(\frac{\pi m}{2l}\right)^2\frac{\sigma_0}{\rho g}(1+\lambda)+\left(\frac{\pi n}{2L}\right)^2\frac{\sigma_0}{\rho g}\right],\ m,n=1,2,3,\ldots \tag{5.2}$$

The coefficient $k_{m,n}$ is given in (5.1). The case $\lambda = 0$ corresponds to zero temperature gradient. An increase in the heating $\lambda > 0$ increases the natural frequencies. For heating according to the equation $f(x) = \lambda\delta(x - \ell/2)$ the coefficient λ in (5.2) is replaced by $-\lambda$. Then an increase in the heating decreases the oscillation frequencies.

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